A. APPENDIX ON NOTATION

In order to be consistent with both the timelike and spacelike conventions on signature, I have taken the Minkowski metric to be

$$\eta_{\mu\nu} = \mathrm{diag}(s, -s\ , -s\ , -s\ ), \tag{1}$$

where $s = +1$ for the timelike convention and $s = -1$ for the spacelike convention. All other sign conventions and conventions on index positions are those of Misner, Thorne and Wheeler [1973].

Unless otherwise noted, Latin indices refer to a coordinate basis,

$$\partial_a = \frac{\partial}{\partial x^a}\ , \quad dx^a\ , \tag{2}$$

while Greek indices refer to either an arbitrary basis or an orthonormal basis,

$$e_\alpha = e_\alpha{}^a\, \partial_a\ , \quad \theta^\alpha = \theta^\alpha{}_a\, dx^a. \tag{3}$$

The vector basis, e_α, and the 1-form basis, θ^α are dual. Hence the matrices $e_\alpha{}^a$ and $\theta^\alpha{}_a$ are inverses:

$$\theta^\alpha{}_a\, e_\alpha{}^b = \delta^b_a\ , \ e_\alpha{}^a\, \theta^\beta{}_a = \delta^\beta_\alpha\ . \tag{4}$$

The coordinate components of the metric, g_{ab}, are related to the components in an arbitrary frame, $g_{\alpha\beta}$, by the formulas

$$g_{ab} = g_{\alpha\beta}\, \theta^\alpha{}_a\ \theta^\beta{}_b, \quad g_{\alpha\beta} = e_\alpha{}^a\, e_\beta{}^b\, g_{ab}. \tag{5}$$

Contracting equation (5a) with g^{bc} and (5b) with $g^{\beta\gamma}$, yields

$$\delta^c_a = \theta^\alpha{}_a\, g_{\alpha\beta}\, \theta^\beta{}_b\, g^{bc}\ , \ \delta^\gamma_\alpha = e_\alpha{}^a\, g^{\gamma\beta}\, e_\beta{}^b\, g_{ba}. \tag{6}$$

Comparison with equations (4) shows that

$$e_\alpha{}^c = g_{\alpha\beta}\, \theta^\beta{}_b\, g^{bc} \quad , \quad \theta^\gamma{}_a = g^{\gamma\beta}\, e_\beta{}^b\, g_{ba}. \tag{7}$$

Taking the determinant of equation (5a) yields

$$\tilde{g} = \hat{g}\ \theta^2 \ , \quad \sqrt{-\tilde{g}} = \sqrt{-\hat{g}}\ \ |\theta|, \tag{8}$$

where

$$\tilde{g} = \det g_{ab} \ , \ \hat{g} = \det g_{\alpha\beta}, \tag{9}$$

$$\theta = \det\ \theta^\alpha{}_a = (\det e_\alpha{}^a)^{-1} \ . \tag{10}$$

I assume that the coordinate basis ∂_a is oriented so that the volume element is

$$\eta = \frac{1}{24}\ \eta_{abcd}\ dx^a \wedge dx^b \wedge dx^c \wedge dx^d$$

$$= \sqrt{-\tilde{g}}\ \ d^4x \ , \tag{11}$$

where η_{abcd} is the totally antisymmetric tensor with

$$\eta_{0123} = \sqrt{-\tilde{g}} \ . \tag{12}$$

If the basis e_α is oriented, so that

$$\theta > 0 \quad , \quad \sqrt{-\tilde{g}} = \sqrt{-\hat{g}}\ .\theta \ , \tag{13}$$

then the volume element may be written as

$$\eta = \frac{1}{24}\, \eta_{\alpha\beta\gamma\delta}\ \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta$$

$$= \sqrt{-\hat{g}}\ \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$$

$$= \sqrt{-\hat{g}}\ \ \theta\ d^4x \ . \tag{14}$$

If the basis e_α, is orthonormal as well as oriented, so that

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(s\ ,\ -s\ ,\ -s\ ,\ -s\ ),$$

$$\hat{g} = -1, \qquad \sqrt{-\tilde{g}} = \theta\ , \tag{15}$$

then the volume element may also be written as

$$\eta = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$$

$$= \theta\ d^4x. \tag{16}$$

For an arbitrary basis, e_α, the commutator functions, $c^\alpha{}_{\beta\gamma}$, are defined by

$$[e_\beta,\ e_\gamma] = c^\alpha{}_{\beta\gamma}\ e_\alpha, \tag{17}$$

and therefore satisfy

$$c^\alpha{}_{\beta\gamma} = \theta^\alpha{}_b(e_\beta{}^c\ \partial_c e_\gamma{}^b - e_\gamma{}^c \partial_c e_\beta{}^b)$$

$$= e_\beta{}^b\ e_\gamma{}^c(\partial_c\theta^\alpha{}_b - \partial_b\theta^\alpha{}_c)\ . \tag{18}$$

The commutator functions, $c^\alpha{}_{\beta\gamma}$, are also called the object of anholonomity since they measure the amount by which the basis, e_α, is not a coordinate (or holonomic) basis.

For an arbitrary covariant derivative, ∇, the connection coefficients, $\Gamma^\alpha{}_{\beta\gamma}$, in an arbitrary basis, e_α, are defined by

$$\nabla_{e_\gamma} e_\beta = \Gamma^\alpha{}_{\beta\gamma}\ e_\alpha. \tag{19}$$

In particular, the coordinate components of the connection coefficients, $\Gamma^a{}_{bc}$, are defined by

$$\nabla_{\partial_c} \partial_b = \Gamma^a{}_{bc}\partial_a\ . \tag{20}$$

These are related to $\Gamma^{\alpha}{}_{\beta\gamma}$ by

$$\Gamma^{\alpha}{}_{\beta\gamma} = \theta^{\alpha}{}_{a}\, e_{\beta}{}^{b}\, e_{\gamma}{}^{c}\, \Gamma^{a}{}_{bc} + \theta^{\alpha}{}_{a}\, e_{\gamma}{}^{c}\, \partial_{c} e_{\beta}{}^{a} \,, \tag{21}$$

$$\Gamma^{a}{}_{bc} = e_{\alpha}{}^{a}\, \theta^{\beta}{}_{b}\, \theta^{\gamma}{}_{c}\, \Gamma^{\alpha}{}_{\beta\gamma} + e_{\alpha}{}^{a}\, \partial_{c} \theta^{\alpha}{}_{b} \,. \tag{22}$$

If e_{α} is an orthonormal frame, then the connection coefficients, $\Gamma^{\alpha}{}_{\beta\gamma}$, are also called the Ricci rotation coefficients. I also find it useful to introduce the mixed components of the connection coefficients, $\Gamma^{\alpha}{}_{\beta c}$, which are defined by

$$\nabla_{\partial_{c}} e_{\beta} = \Gamma^{\alpha}{}_{\beta c}\, e_{\alpha} \,. \tag{23}$$

These are related to $\Gamma^{\alpha}{}_{\beta\gamma}$ and $\Gamma^{a}{}_{bc}$ by

$$\Gamma^{\alpha}{}_{\beta\gamma} = e_{\gamma}{}^{c}\, \Gamma^{\alpha}{}_{\beta c}, \tag{24}$$

$$\Gamma^{\alpha}{}_{\beta c} = \theta^{\gamma}{}_{c}\, \Gamma^{\alpha}{}_{\beta\gamma} \,, \tag{25}$$

$$\Gamma^{a}{}_{bc} = e_{\alpha}{}^{a}\, \theta^{\beta}{}_{b}\, \Gamma^{\alpha}{}_{\beta c} + e_{\alpha}{}^{a}\, \partial_{c} \theta^{\alpha}{}_{b}, \tag{26}$$

$$\Gamma^{\alpha}{}_{\beta c} = \theta^{\alpha}{}_{a}\, e_{\beta}{}^{b}\, \Gamma^{a}{}_{bc} + \theta^{\alpha}{}_{a}\, \partial_{c} e_{\beta}{}^{a}. \tag{27}$$

The torsion tensor, Q, of the connection, ∇, is the vector valued operator

$$Q(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] . \tag{28}$$

Its components in an arbitrary basis, e_α, are

$$Q^\alpha{}_{\beta\gamma} = \theta^\alpha(Q(e_\beta, e_\gamma))$$

$$= \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma} - c^\alpha{}_{\beta\gamma} \tag{29}$$

$$= e_\beta{}^b\, \Gamma^\alpha{}_{\gamma b} - e_\gamma{}^c\, \Gamma^\alpha{}_{\beta c} + e_\beta{}^b\, e_\gamma{}^c (\partial_b \theta^\alpha{}_c - \partial_c \theta^\alpha{}_b). \tag{30}$$

In particular, the coordinate components are

$$Q^a{}_{bc} = dx^a(Q(\partial_b, \partial_c))$$

$$= \Gamma^a{}_{cb} - \Gamma^a{}_{bc} \tag{31}$$

$$= e_\alpha{}^a\, (\theta^\gamma{}_c\, \Gamma^\alpha{}_{\gamma b} - \theta^\beta{}_b\, \Gamma^\alpha{}_{\beta c} + \partial_b \theta^\alpha{}_c - \partial_c \theta^\alpha{}_b). \tag{32}$$

The mixed components of the torsion are defined as

$$Q^\alpha{}_{bc} = \theta^\alpha(Q(\partial_b, \partial_c))$$

$$= \theta^\alpha{}_a\, (\Gamma^a{}_{cb} - \Gamma^a{}_{bc}) \tag{33}$$

$$= \theta^\gamma{}_c\, \Gamma^\alpha{}_{\gamma b} - \theta^\beta{}_b\, \Gamma^\alpha{}_{\beta c} + \partial_b \theta^\alpha{}_c - \partial_c \theta^\alpha{}_b. \tag{34}$$

If $Q = 0$, then the connection is called torsion-free. Since by (31), the coordinate components of a torsion-free connection, $\Gamma^a{}_{bc}$, are symmetric in b and c, a torsion-free connection is also called a symmetric connection.

In an arbitrary basis, e_α, the covariant derivative of the metric is

$$\nabla_\gamma g_{\alpha\beta} = e_\gamma g_{\alpha\beta} - \Gamma^\delta{}_{\alpha\gamma}\, g_{\delta\beta} - \Gamma^\delta{}_{\beta\gamma}\, g_{\alpha\delta} . \tag{35}$$

In particular, the coordinate components are

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d{}_{ac}\, g_{db} - \Gamma^d{}_{bc}\, g_{ad}, \tag{36}$$

and the orthonormal components are

$$\nabla_\gamma g_{\alpha\beta} = - \Gamma^\delta{}_{\alpha\gamma} g_{\delta\beta} - \Gamma^\delta{}_{\beta\gamma}\, g_{\alpha\delta}. \tag{37}$$

The mixed components of the covariant derivative of the metric are defined as

$$\nabla_c g_{\alpha\beta} = \partial_c g_{\alpha\beta} - \Gamma^\delta{}_{\alpha c}\, g_{\delta\beta} - \Gamma^\delta{}_{\beta c}\, g_{\alpha\delta}, \tag{38}$$

which for orthonormal frames becomes

$$\nabla_c g_{\alpha\beta} = - \Gamma^\delta{}_{\alpha c}\, g_{\delta\beta} - \Gamma^\delta{}_{\beta c}\, g_{\alpha\delta} . \tag{39}$$

If $\nabla g = 0$, then the connection is called metric-compatible or metric or compatible with the metric. The condition $\nabla g = 0$ is called the metric-compatibility condition. If there exists a 1-form, $\lambda = \lambda_\gamma\, \theta^\gamma = \lambda_c dx^c$, such that

$$\nabla_\gamma g_{\alpha\beta} = - \frac{1}{2}\, \lambda_\gamma\, g_{\alpha\beta}, \tag{40}$$

then the connection is called Weyl-compatible or semi-metric. The condition (40) is called the Weyl-compatibility condition and the 1-form λ is called the Weyl potential. Some authors use a different numerical factor in (40). I use $-\frac{1}{2}$ so that λ_γ comes out as the trace, $\lambda_\gamma = \lambda^\alpha{}_{\alpha\gamma}$, of the defect tensor, $\lambda^\alpha{}_{\beta\gamma}$, defined below.

The connection coefficients, $\Gamma^{\alpha}{}_{\beta\gamma}$, in an arbitrary frame, e_{α}, can be expressed in terms of the partial derivatives of the components of the metric $e_{\delta}g_{\beta\gamma}$; the commutator functions, $c_{\delta\beta\gamma} = g_{\delta\alpha}\, c^{\alpha}{}_{\beta\gamma}$; the torsion $Q_{\delta\beta\gamma} = g_{\delta\alpha}\, Q^{\alpha}{}_{\beta\gamma}$; and the covariant derivative of the metric, $\nabla_{\delta}g_{\beta\gamma}$:

$$\begin{aligned}
\Gamma^{\alpha}{}_{\beta\gamma} = {} & \frac{1}{2}\, g^{\alpha\delta}(e_{\beta}g_{\delta\gamma} + e_{\gamma}g_{\delta\beta} - e_{\delta}g_{\beta\gamma}) \\
& + \frac{1}{2}\, g^{\alpha\delta}(c_{\beta\delta\gamma} + c_{\gamma\delta\beta} - c_{\delta\beta\gamma}) \\
& + \frac{1}{2}\, g^{\alpha\delta}(Q_{\beta\delta\gamma} + Q_{\gamma\delta\beta} - Q_{\delta\beta\gamma}) \\
& - \frac{1}{2}\, g^{\alpha\delta}(\nabla_{\beta}g_{\delta\gamma} + \nabla_{\gamma}g_{\delta\beta} - \nabla_{\delta}g_{\beta\gamma}) \; .
\end{aligned} \tag{41}$$

This formula is derived by substituting equation (29) for Q and equation (35) for ∇g into the right hand side of (41). Notice that the index pattern is the same on each line and that the signs are the same on each line except for the overall minus on the last line.

The unique metric-compatible and torsion-free connection is called the Christoffel connection and has the connection coefficients

$$\begin{aligned}
\{^{\alpha}{}_{\beta\gamma}\} = {} & \frac{1}{2}\, g^{\alpha\delta}(e_{\beta}g_{\delta\gamma} + e_{\gamma}g_{\delta\beta} - e_{\delta}g_{\beta\gamma}) \\
& + \frac{1}{2}\, g^{\alpha\delta}(c_{\beta\delta\gamma} + c_{\gamma\delta\beta} - c_{\delta\beta\gamma}).
\end{aligned} \tag{42}$$

(Note that many authors use the symbol $\{^{\alpha}{}_{\beta\gamma}\}$ only when the basis is a coordinate basis. I use it for an arbitrary basis.) The metric-compatibility condition says

$$e_{\gamma}g_{\alpha\beta} = g_{\alpha\delta}\{^{\delta}{}_{\beta\gamma}\} + g_{\beta\delta}\{^{\delta}{}_{\alpha\gamma}\}, \tag{43}$$

while the torsion-free condition says

$$c^{\alpha}{}_{\beta\gamma} = \{^{\alpha}{}_{\gamma\beta}\} - \{^{\alpha}{}_{\beta\gamma}\} \quad . \tag{44}$$

Thus in an orthonormal basis, $g_{\alpha\delta}\{{}^{\delta}{}_{\beta\gamma}\}$ is antisymmetric in α and β. On the other hand, in a coordinate basis $\{{}^{a}{}_{bc}\}$ is symmetric in b and c.

The collection of terms,

$$\{c\}^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\delta}(c_{\beta\delta\gamma} + c_{\gamma\delta\beta} - c_{\delta\beta\gamma}) \ , \tag{45}$$

measures the amount by which the basis, e_{α}, is not a coordinate (or holonomic) basis and so might be called the anholonomity symbol to distinquish it from the object of anholonomity which is $c^{\alpha}{}_{\beta\gamma}$ itself. Equation (45) may be inverted to give

$$c^{\alpha}{}_{\beta\gamma} = \{c\}^{\alpha}{}_{\gamma\beta} - \{c\}^{\alpha}{}_{\beta\gamma} \ . \tag{46}$$

The collection of terms,

$$\{eg\}^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\delta}(e_{\beta}g_{\delta\gamma} + e_{\gamma}g_{\delta\beta} - e_{\delta}g_{\beta\gamma}), \tag{47}$$

measures the amount by which the basis, e_{α}, is not an orthonormal basis (or at least the amount by which the components of the metric, $g_{\beta\gamma}$, are non-constant) and so might be called the anormality symbol, while $e_{\delta}g_{\beta\gamma}$ might be called the object of anormality. Equation (47) may be inverted to give

$$e_{\gamma}g_{\alpha\beta} = \{eg\}_{\alpha\beta\gamma} + \{eg\}_{\beta\alpha\gamma}. \tag{48}$$

Combining (42), (45) and (47) shows that the Christoffel symbol,

$$\{{}^{\alpha}{}_{\beta\gamma}\} = \{eg\}^{\alpha}{}_{\beta\gamma} + \{c\}^{\alpha}{}_{\beta\gamma} \quad , \tag{49}$$

is the sum of the anholonomity symbol and the anormality symbol.

The difference between a general connection, $\Gamma^{\alpha}{}_{\beta\gamma}$, and the Christoffel connection, $\{^{\alpha}{}_{\beta\gamma}\}$, is the defect tensor,

$$\lambda^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \{^{\alpha}{}_{\beta\gamma}\} \tag{50}$$

$$= \frac{1}{2} g^{\alpha\delta}(Q_{\beta\delta\gamma} + Q_{\gamma\delta\beta} - Q_{\delta\beta\gamma}) - \frac{1}{2} g^{\alpha\delta}(\nabla_{\beta}g_{\delta\gamma} + \nabla_{\gamma}g_{\delta\beta} - \nabla_{\delta}g_{\beta\gamma}) , \tag{51}$$

which may itself be decomposed as the sum,

$$\lambda^{\alpha}{}_{\beta\gamma} = - K^{\alpha}{}_{\beta\gamma} - M^{\alpha}{}_{\beta\gamma} , \tag{52}$$

of the negative of the contorsion tensor,

$$K^{\alpha}{}_{\beta\gamma} = - \frac{1}{2} g^{\alpha\delta}(Q_{\beta\delta\gamma} + Q_{\gamma\delta\beta} - Q_{\delta\beta\gamma}), \tag{53}$$

and the negative of the non-metricity tensor,

$$M^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(\nabla_{\beta}g_{\delta\gamma} + \nabla_{\gamma}g_{\delta\beta} - \nabla_{\delta}g_{\beta\gamma}). \tag{54}$$

Equations (52), (53) and (54) may be solved for

$$Q^{\alpha}{}_{\beta\gamma} = K^{\alpha}{}_{\beta\gamma} - K^{\alpha}{}_{\gamma\beta} = \lambda^{\alpha}{}_{\gamma\beta} - \lambda^{\alpha}{}_{\beta\gamma}, \tag{55}$$

$$\nabla_{\gamma}g_{\alpha\beta} = M_{\alpha\beta\gamma} + M_{\beta\alpha\gamma} = - \lambda_{\alpha\beta\gamma} - \lambda_{\beta\alpha\gamma} . \tag{56}$$

Combining (49), (50) and (52) shows that a general connection may be written as

$$\Gamma^{\alpha}{}_{\beta\gamma} = \{^{\alpha}{}_{\beta\gamma}\} + \lambda^{\alpha}{}_{\beta\gamma} \tag{57}$$

$$= \{eg\}^{\alpha}{}_{\beta\gamma} + \{c\}^{\alpha}{}_{\beta\gamma} - K^{\alpha}{}_{\beta\gamma} - M^{\alpha}{}_{\beta\gamma} . \tag{58}$$

A metric-compatible connection with arbitrary torsion is called a Cartan connection. Thus a Cartan connection satisfies

$$\nabla_\alpha g_{\beta\gamma} = 0 \quad , \quad M^\alpha{}_{\beta\gamma} = 0, \tag{59}$$

$$\lambda^\alpha{}_{\beta\gamma} = - K^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(Q_{\beta\delta\gamma} + Q_{\gamma\delta\beta} - Q_{\delta\beta\gamma}). \tag{60}$$

A torsion-free Weyl-compatible connection is called a Weyl connection. Thus a Weyl connection satisfies

$$Q^\alpha{}_{\beta\gamma} = 0 \quad , \quad K^\alpha{}_{\beta\gamma} = 0 , \tag{61}$$

$$\nabla_\alpha g_{\beta\gamma} = - \frac{1}{2} \lambda_\alpha \, g_{\beta\gamma} \quad , \quad \lambda_\alpha = \lambda^\beta{}_{\beta\alpha} \quad , \tag{62}$$

$$\lambda^\alpha{}_{\beta\gamma} = - M^\alpha{}_{\beta\gamma} = \frac{1}{4} (\lambda_\beta \delta^\alpha_\gamma + \lambda_\gamma \delta^\alpha_\beta - \lambda^\alpha g_{\beta\gamma}). \tag{63}$$

A Weyl-compatible connection with arbitrary torsion is called a Weyl-Cartan connection. A connection with arbitrary torsion and arbitrary covariant derivative of the metric is referred to as a general connection. To be generic, I use the term full connection to refer to any connection (other than the Christoffel connection) which may have some set of restrictions on the torsion and the covariant derivative of the metric which I do not care to specify.

The full Riemann curvature tensor, $\hat{R}$, of the full connection, ∇, is the vector valued operator

$$\hat{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ . \tag{64}$$

Its components in an arbitrary basis, e_α, are

$$\begin{aligned} \hat{R}^{\alpha}{}_{\beta\gamma\delta} &= \theta^{\alpha}(\hat{R}(e_\gamma, e_\delta) e_\beta) \\ &= e_\gamma \Gamma^{\alpha}{}_{\beta\delta} - e_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\varepsilon\gamma} \Gamma^{\varepsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\varepsilon\delta} \Gamma^{\varepsilon}{}_{\beta\gamma} - c^{\varepsilon}{}_{\gamma\delta} \Gamma^{\alpha}{}_{\beta\varepsilon} \ . \end{aligned} \tag{65}$$

In particular, the coordinate components are

$$\begin{aligned} \hat{R}^{a}{}_{bcd} &= dx^{a}(\hat{R}(\partial_c, \partial_d) \partial_b) \\ &= \partial_c \Gamma^{a}{}_{bd} - \partial_d \Gamma^{a}{}_{bc} + \Gamma^{a}{}_{ec} \Gamma^{e}{}_{bd} - \Gamma^{a}{}_{ed} \Gamma^{e}{}_{bc} \ . \end{aligned} \tag{66}$$

The mixed components of the full curvature are defined as

$$\begin{aligned} \hat{R}^{\alpha}{}_{\beta cd} &= \theta^{\alpha}(\hat{R}(\partial_c, \partial_d) e_\beta) \\ &= \partial_c \Gamma^{\alpha}{}_{\beta d} - \partial_d \Gamma^{\alpha}{}_{\beta c} + \Gamma^{\alpha}{}_{\varepsilon c} \Gamma^{\varepsilon}{}_{\beta d} - \Gamma^{\alpha}{}_{\varepsilon d} \Gamma^{\varepsilon}{}_{\beta c} \ . \end{aligned} \tag{67}$$

By contracting the full Riemann curvature one obtains the full Ricci curvature (asymmetric),

$$\hat{R}_{\beta\delta} = \hat{R}^{\gamma}{}_{\beta\gamma\delta} = e_\gamma{}^{c} e_\delta{}^{d} \hat{R}^{\gamma}{}_{\beta cd} \ , \tag{68}$$

the full scalar curvature,

$$\hat{R} = g^{\beta\delta} \hat{R}_{\beta\delta} \ , \tag{69}$$

and the full Einstein curvature (asymmetric),

$$\hat{G}_{\beta\delta} = \hat{R}_{\beta\delta} - \frac{1}{2} g_{\beta\delta} \hat{R}. \tag{70}$$

I have put a caret (^) over the full curvature tensors to distinguish them from the corresponding quantities for the Christoffel connection which carry a tilde (~). Thus in an arbitrary basis, e_α, the Christoffel Riemann curvature has the components

$$\tilde{R}^\alpha{}_{\beta\gamma\delta} = e_\gamma\{^\alpha{}_{\beta\delta}\} - e_\delta\{^\alpha{}_{\beta\gamma}\} + \{^\alpha{}_{\varepsilon\gamma}\}\{^\varepsilon{}_{\beta\delta}\} - \{^\alpha{}_{\varepsilon\delta}\}\{^\varepsilon{}_{\beta\gamma}\} - c^\varepsilon{}_{\gamma\delta}\{^\alpha{}_{\beta\varepsilon}\} . \tag{71}$$

In a coordinate basis, the components are

$$\tilde{R}^a{}_{bcd} = \partial_c\{^a{}_{bd}\} - \partial_d\{^a{}_{bc}\} + \{^a{}_{ec}\}\{^e{}_{bd}\} - \{^a{}_{ed}\}\{^e{}_{bc}\} . \tag{72}$$

Finally the mixed components are

$$\tilde{R}^\alpha{}_{\beta cd} = \partial_c\{^\alpha{}_{\beta d}\} - \partial_d\{^\alpha{}_{\beta c}\} + \{^\alpha{}_{\varepsilon c}\}\{^\varepsilon{}_{\beta d}\} - \{^\alpha{}_{\varepsilon d}\}\{^\varepsilon{}_{\beta c}\} . \tag{73}$$

The Christoffel Ricci curvature (symmetric) is

$$\tilde{R}_{\beta\delta} = \tilde{R}^\gamma{}_{\beta\gamma\delta} = e_\gamma{}^c\, e_\delta{}^d\, \tilde{R}^\gamma{}_{\beta cd} . \tag{74}$$

The Christoffel scalar curvature is

$$\tilde{R} = g^{\beta\delta}\, \tilde{R}_{\beta\delta} , \tag{75}$$

and the Christoffel Einstein curvature (symmetric) is

$$\tilde{G}_{\beta\delta} = \tilde{R}_{\beta\delta} - \frac{1}{2}\, g_{\beta\delta}\, \tilde{R}. \tag{76}$$